October Math Gems

PROBLEM OF THE WEEK 17

§1 problems

Problem 1.1. Solve for x

$$\sqrt[4]{1-x^2}+\sqrt[4]{1-x}+\sqrt[4]{1+x}=3$$

answer.

$$((1-x)(1+x))^{\frac{1}{4}}+(1-x)^{\frac{1}{4}}+(1+x)^{\frac{1}{4}}=3$$

Let,

$$a=(1-x)^{\frac{1}{4}}, \qquad b=(1+x)^{\frac{1}{4}}$$

$$ab+a+b+1=3+1$$

$$(1+b)(1+a)=4 \implies a=1 \quad \text{and} \quad b=1$$

Now, we can say that $1+x=1-x$. So, $x=0$. □

Problem 1.2. Find all points (x,y) where the functions $f(x),g(x),h(x)$ have the same value:

$$f(x)=2^{x-5}+3, \qquad g(x)=2x-5, \qquad h(x)=\frac{8}{x}+10$$

answer.

$$f(x)=g(x)=h(x)$$

$$2^{x-5}+3=2x-5=\frac{8}{x}+10$$

For,

$$2x-5=\frac{8}{x}+10 \qquad \text{(multiply by } x\text{)}$$

$$2x^2-5x=8+10x \implies 2x^2-15x-8=0$$

$$x=8 \qquad \text{and} \qquad x=\frac{-1}{2}$$

For,

$$2^{x-5}+3=2x-5$$

will give us an integer solution $x=8$. we can verify solution $x=8$ by plugging the value 8 in the place of x. So, we will see that

$$f(x)=g(x)=h(x)=11 \qquad \text{As } (x=8)$$

So, the points that $f(x)=g(x)=h(x)$ have the same value is $(8,11)$. □

Problem 1.3. Solve for x

$$(12x-1)(6x-1)(4x-1)(3x-1)=5$$

answer. Let $x=\frac{y}{12}$,

$$(12\times\frac{y}{12}-1)(6\times\frac{y}{12}-1)(4\times\frac{y}{12}-1)(3\times\frac{y}{12}-1)=5$$

$$(y-1)(\frac{y}{2}-1)(\frac{y}{3}-1)(\frac{y}{4}-1)=5$$

Multiply this equation by $2\times 3\times 4$, So we get

$$(y-1)(y-2)(y-3)(y-4)=120$$

Now, we can solve for the four roots by knowing the properties of the 4^{th} degree equation

$$y=\frac{5\pm\sqrt{-39}}{2} \qquad y=6 \qquad y=-1$$

Do not forget that we assume before that $x=\frac{y}{12}$. So,

$$x=\frac{5\pm\sqrt{-39}}{24} \qquad x=\frac{1}{2} \qquad x=\frac{-1}{12}$$

□

Problem 1.4. Solve

$$x^{[\frac{3}{4}(\log(x))^2+(\log(x))-\frac{5}{4}]}=\sqrt{2}$$

answer.

$$\log_2 x^{[\frac{3}{4}(\log(x))^2+(\log(x))-\frac{5}{4}]}=\log_2(2)^{\frac{1}{2}}$$

$$[\frac{3}{4}(\log_2(x))^2+(\log_2(x))-\frac{5}{4}]\times\log_2 x=\frac{1}{2}$$

Let,

$$\log_2 x=y$$

So, we get

$$[\frac{3}{4}(y)^2+(y)-\frac{5}{4}]\times y=\frac{1}{2} \implies 3y^3+4y^2-5y-2=0$$

$$3y^3+4y^2-5y-2=(y-1)(3y+1)(y+2)=0$$

$$y=1 \qquad y=-2 \qquad y=\frac{-1}{3}$$

Now,

$$\log_2 x=1 \qquad \log_2 x=-2 \qquad \log_2 x=\frac{-1}{3}$$

So, we will get three values of x,

$$x=2^1=2 \qquad x=2^{-2}=\frac{1}{4} \qquad x=2^{\frac{-1}{3}}$$

□

Problem 1.5. Solve for x

$$\frac{2x}{2x^2-5x+3}+\frac{13x}{2x^2+x+3}=6$$

answer. First, divide the numerator and denominator by x, As $x = 0$ is not one of the roots of this equation. So,

$$\frac{2}{2x - 5 + \frac{3}{x}} + \frac{13}{2x + 1 + \frac{3}{x}} = 6$$

Now, Let

$$a = 2x + \frac{3}{x}$$

$$\frac{2}{a-5} + \frac{13}{a+1} = 6 \implies 13(a-5) + 2(a+1) = 6(a-5)(a+1)$$

$$13a - 65 + 2a + 2 = 6(a^2 - 4a - 5) \implies 6a^2 - 39a + 33 = 0$$

Now, we can solve for a

$$a = \begin{cases} \frac{11}{2} \\ 1 \end{cases}$$

For $a = 1$, there is no real solutions!

For $a = \frac{11}{2}$,

$$(4x - 3)(x - 2) = 0 \implies x = \frac{3}{4}, \qquad x = 2$$

So, the zeros of this equation is $\{\frac{3}{4}, 2\}$ □

Problem 1.6. If $x^2 + y^2 = 4$, Find the largest value of $3x + 4y$.

answer.

$$\because (ax + by)^2 + (bx - ay)^2 = (a^2 + b^2)(x^2 + y^2)$$

$$\therefore (3x + 4y)^2 + (4x - 3y)^2 = (9 + 16)(x^2 + y^2) = 25 \times 4 = 100$$

$$\because (3x + 4y)^2 = 100 - (4x - 3y)^2 \qquad \text{and} \qquad (4x - 3y)^2 \geq 0$$

$$\therefore |3x + 4y| \leq 10 \implies -10 \leq 3x + 4y \leq 10$$

so the greatest value of $3x + 4y$ is 10. □

Problem 1.7. If $ax + (b - 3) = (5a - 1)x + 3b$ has more than one solution, find the value of $100a + 4b$.

answer.

$$x(a - 5a + 1) = 3b - b + 3 \implies (1 - 4a)x = 2b + 3$$

We can ask ourselves When the equation has more than one solution?

$$0 \times x = 0 \qquad \text{(x is infinite)}$$

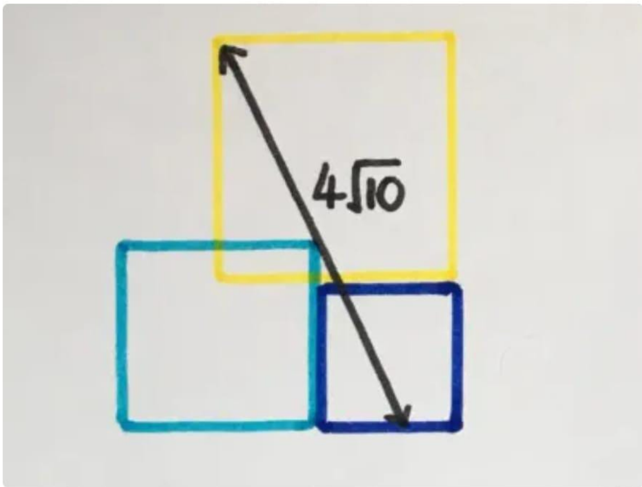
Now, we can use this idea in this proof

$$(1 - 4b) = 0 \implies a = \frac{1}{4} \qquad 2b + 3 = 0 \implies b = \frac{-3}{2}$$

$$100a + 4b = 100 \times \frac{1}{4} + 4 \times \frac{-3}{2} = 25 - 6 = 19$$

□

Problem 1.8. The side lengths of the three squares are consecutive integers. What's the total area?



answer. Note the yellow square is a two-unit wider than the blue one.

$$16 + x^2 = (4\sqrt{10})^2 = 160$$

which means the squares have lengths 5, 6, and 7. □

Problem 1.9. If $6^{-z} = 2^x = 3^y$ then the value of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

is ?

answer.

$$6^{-z} = 2^x = 3^y = a$$

So,

$$2^x = a \implies 2 = \sqrt[x]{a} \implies 2 = a^{\frac{1}{x}}$$

and so on with others so we get

$$2 = a^{\frac{1}{x}} \qquad 6 = a^{\frac{-1}{z}} \qquad 3 = a^{\frac{1}{y}}$$

We can see that

$$2 \times 3 = a^{\frac{1}{x}} \times a^{\frac{1}{y}} = a^{\frac{1}{x}+\frac{1}{y}} = a^{\frac{-1}{z}}$$

Now, we can say that

$$\frac{1}{x} + \frac{1}{y} = \frac{-1}{z}$$

So,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{-1}{z} + \frac{1}{z} = 0$$

□

Problem 1.10. Find the solution set of the equation

$$3\cos^{-1}x = \sin^{-1}(\sqrt{1-x^2}\times(4x^2-1))$$

answer.

$$y = \sin^{-1}(\sqrt{1-x^2}\times(4x^2-1))$$

$$x = \cos\theta \implies \theta = \cos^{-1}x$$

$$y = \sin^{-1}(\sin\theta(4\cos^2\theta-1)) = \sin^{-1}(\sin(3\theta)) = 3\theta = 3\cos^{-1}(x) \qquad (\frac{-\pi}{6}\le\theta\le\frac{\pi}{6})$$

also,

$$0\le\theta\le\frac{\pi}{6} \implies 1\ge x\ge\frac{\sqrt{3}}{2}$$

So the solution set

$$x\in[\frac{\sqrt{3}}{2},1]$$

□

Problem 1.11. if p, q are odd positive numbers since

$$(1+3+5+\cdots+p)+(1+3+5+\cdots+q) = (1+3+5+\cdots+19)$$

Find the value of $p+q$.

answer. the rule of the sum of the odd numbers

$$1+3+5+\cdots+(2m-1) = m^2$$

$$2m-1 = p \implies 2m = p+1 \implies m = \frac{p+1}{2}$$

$$\therefore 1+3+5+7+\cdots+p = (\frac{p+1}{2})^2 \to (1)$$

$$\therefore 1+3+5+7+\cdots+q = (\frac{q+1}{2})^2 \to (2)$$

$$\therefore 1+3+5+7+\cdots+19 = (\frac{19+1}{2})^2 = 100 \to (3)$$

$$(\frac{q+1}{2})^2+(\frac{p+1}{2})^2 = 100$$

there is one possible solution that is

$$6^2+8^2 = 10^2$$

$$p+q = 11+15 = 26$$

□

Problem 1.12. Of 450 students assembled for a concert, 40 percent were boys. After a bus containing an equal number of boys and girls brought more students to the concert, 41 percent of the students at the concert were boys. Find the number of students on the bus.

answer. Of the 450 students originally at the concert, $(0.40)450 = 180$ were boys. Let n be the number of boys on the bus, so the number of students on the bus was $2n$. Then $0.41 = \frac{180+n}{450+2n}$ which simplifies to $(0.41)450 + 0.82n = 180 + n$ and $n = \frac{0.41\times 450-180}{1-0.82} = \frac{4.5}{0.18} = 25$. Thus, the number of students on the bus was $2 \times 25 = 50$ □

Problem 1.13. Let a be a positive real number such that

$$4a^2 + \frac{1}{a^2} = 117.$$

Find

$$8a^3 + \frac{1}{a^3}.$$

answer. Adding 4 to each side of the given equation yields

$$4a^2 + 4 + \frac{1}{a^2} = 121 = 11^2,$$

so $2a + \frac{1}{a} = 11$. Hence, cubing both sides yields

$$11^3 = 8a^3 + 3 \times 4a + 3 \times \frac{2}{a} + \frac{1}{a^3} = 8a^3 + \frac{1}{a^3} + 6(2a + \frac{1}{a}) = 8a3 + \frac{1}{a^3} + 6 \times 11.$$

It follows that

$$8a^3 + \frac{1}{a^3} = 11^3 - 6 \times 11 = 1331 - 66 = 1265.$$

□

Problem 1.14. Find $x^6 + \frac{1}{x^6}$ if $x + \frac{1}{x} = 3$.

answer.

$$\therefore (x + \frac{1}{x})^2 = 9$$

$$\therefore x^2 + (\frac{1}{x})^2 = 7$$

$$\therefore (x^2 + (\frac{1}{x})^2)^3 = (7)^3$$

$$\therefore x^6 + (\frac{1}{x})^6 + 3(x^2 + \frac{1}{x^2}) = (7)^3$$

$$\therefore x^6 + (\frac{1}{x})^6 = (7)^3 - 3(7) = 322$$

□

Problem 1.15. Let $a_1 = 2021$ and for $n \geq 1$ let $a_{n+1} = \sqrt{4 + a_n}$. Then a_5 can be written as

$$\sqrt{\frac{m + \sqrt{n}}{2}} + \sqrt{\frac{m - \sqrt{n}}{2}}$$

where m and n are positive integers. Find $10m + n$.

answer. One can calculate

$$\begin{aligned} a_2 &= \sqrt{2021+4} = \sqrt{2025} = 45 \\ a_3 &= \sqrt{45+4} = \sqrt{49} = 7 \\ a_4 &= \sqrt{7+4} = \sqrt{11} \\ a_5 &= \sqrt{\sqrt{11}+4}. \end{aligned}$$

Then m and n must satisfy

$$(\sqrt{\sqrt{11}+4})^2 = \left(\sqrt{\frac{m+\sqrt{n}}{2}} + \sqrt{\frac{m-\sqrt{n}}{2}}\right)^2$$

so

$$\begin{aligned} 4+\sqrt{11} = \frac{m+\sqrt{n}}{2} + 2\left(\sqrt{\frac{m+\sqrt{n}}{2}}\right)\left(\sqrt{\frac{m-\sqrt{n}}{2}}\right) + \frac{m-\sqrt{n}}{2} \\ = m+\sqrt{m^2-n}. \end{aligned}$$

It follows that $m=4$ and $n=5$. The requested sum is $10\cdot 4+5=45$. □

Problem 1.16. The product

$$\left(\frac{1+1}{1^2+1}+\frac{1}{4}\right)\left(\frac{2+1}{2^2+1}+\frac{1}{4}\right)\left(\frac{3+1}{3^2+1}+\frac{1}{4}\right)\cdots\left(\frac{2022+1}{2022^2+1}+\frac{1}{4}\right)$$

can be written as $\frac{q}{2^r\cdot s}$, where r is a positive integer, and q and s are relatively prime odd positive integers. Find s.

answer. Note that $\frac{n+1}{n^2+1}+\frac{1}{4}=\frac{1}{4}\cdot\frac{(n+2)^2+1}{n^2+1}$. Thus, the given product telescopes and is equal to

$$\frac{1}{4^{2022}}\cdot\frac{1}{1^2+1}\cdot\frac{1}{2^2+1}\cdot\left(2023^2+1\right)\left(2024^2+1\right) = \frac{\left(2023^2+1\right)\left(2024^2+1\right)}{2^{4045}\cdot 5}.$$

The last digit of 2023^2 must be 9 , so 2023^2+1 is a multiple of 5 . It follows that the denominator of the reduced fraction is a power of 2 , and the value of s is 1 □

Problem 1.17. Let a and b be positive real numbers satisfying

$$\frac{a}{b}\left(\frac{a}{b}+2\right)+\frac{b}{a}\left(\frac{b}{a}+2\right)=2022.$$

Find the positive integer n such that

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}=\sqrt{n}$$

answer. Adding 3 to both sides of the given equation yields

$$\left(\frac{a}{b}+\frac{b}{a}+1\right)^2=2025,$$

which implies that $\frac{a}{b}+\frac{b}{a}+1=45$. Then $\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2=\frac{a}{b}+2+\frac{b}{a}=46$. In particular, the original equation is satisfied by $a=22+\sqrt{483}$ and $b=1$. □

Problem 1.18. Let a be a real number such that

$$5\sin^4\left(\frac{a}{2}\right)+12\cos a=5\cos^4\left(\frac{a}{2}\right)+12\sin a.$$

There are relatively prime positive integers m and n such that $\tan a=\frac{m}{n}$. Find $10m+n$.

answer. Rewrite the given equation as

$$12\cos a-12\sin a=5\cos^4\left(\frac{a}{2}\right)-5\sin^4\left(\frac{a}{2}\right)$$

Then

$$12\cos a-12\sin a=5\left[\cos^2\left(\frac{a}{2}\right)-\sin^2\left(\frac{a}{2}\right)\right]=5\cos a$$

It follows that $\tan a=\frac{\sin a}{\cos a}=\frac{7}{12}$. The requested expression is $10\cdot 7+12=82$. □

Problem 1.19. The sum of the solutions to the equation

$$x^{\log_2 x}=\frac{64}{x}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

answer. Taking the logarithm base 2 of both sides of the equation gives $\log_2\left(x^{\log_2 x}\right)=\log_2\left(\frac{64}{x}\right)$, from which $(\log_2 x)(\log_2 x)=6-\log_2 x$.
This is equivalent to $(\log_2 x-2)(\log_2 x+3)=0$, whose solutions are 4 and $\frac{1}{8}$.
The sum of the solutions is $\frac{33}{8}$. The requested sum is $33+8=41$. □

Problem 1.20. The length of the perimeter of a right triangle is 60 inches and the length of the altitude perpendicular to the hypotenuse is 12 inches. Find the sides of the triangle.

answer. Let a, b, and c denote the sides, the last being the hypotenuse. The three parts of the condition are expressed by

$$\begin{aligned} a+b+c&=60\\ a^2+b^2&=c^2\\ ab&=12\text{c}. \end{aligned}$$

Observing that

$$(a+b)^2=a^2+b^2+2ab$$

we obtain

$$(60-c)^2=c^2+24c.$$

Hence $c=25$ and either $a=15, b=20$ or $a=20, b=15$ (no difference for the triangle). □